Improved Private Information Retrieval Rate for Noncolluding Coded Distributed Storage

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Joint work with S. Kumar, E. Rosnes, and A. Graell i Amat

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Key ingredient in BigData and IoT: Distributed Storage

- Distributed storage using coding techniques:[1]
  - data is encoded by an $[n, k]$ linear code, and then distributed and stored across n storage servers

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- Coded DSSs can be made reliable, robust, efficient, and secure[2]
  - E.g., locally repairable codes (LRCs)

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Coded Data in DSSs

Data consists of M files, file index $m \in \{1, \ldots, M\} \triangleq [1 : M]$, and each file $\mathbf{X}^{(m)}$ has size/length β (file size or subpacketization)

Coded Data in DSSs

$$\mathbf{X}^{(m)}: \ \beta = \beta' k$$

$$\beta' \left\{ \underbrace{\begin{array}{|c|c|c|c|} \hline X_{1,1}^{(m)} & X_{1,2}^{(m)} & \cdots & X_{1,k}^{(m)} \\ \hline X_{2,1}^{(m)} & X_{2,2}^{(m)} & \cdots & X_{2,k}^{(m)} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline X_{\beta',1}^{(m)} & X_{\beta',2}^{(m)} & \cdots & X_{\beta',k}^{(m)} \\ \hline \end{array}}_{k} \right.$$

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\hline
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\hline
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\hline
\vdots & \vdots & \cdots & \vdots \\
\hline
X^{(m)}_{\beta',1} & X^{(m)}_{\beta',2} & \cdots & X^{(m)}_{\beta',k} \\
\hline
\end{array}}_{k} \right.$$

a linear $[n, k]$ storage code $\mathscr{C}$

$C^{(m)}: \beta = \beta' \times n$

$$\begin{array}{|c|c|c|c|}
\hline
C^{(m)}_{1,1} & C^{(m)}_{1,2} & \cdots & C^{(m)}_{1,n} \\
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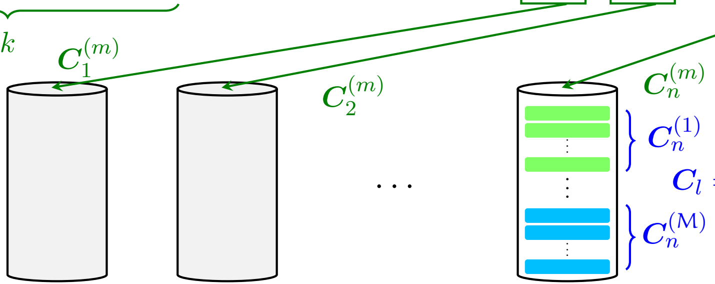

$$C_l \triangleq \begin{pmatrix} C_l^{(1)} \\ \vdots \\ C_l^{(M)} \end{pmatrix}$$

Coded Data in DSSs

- Replicated data:
  - using an $[n, 1]$ repetition code to encode the data
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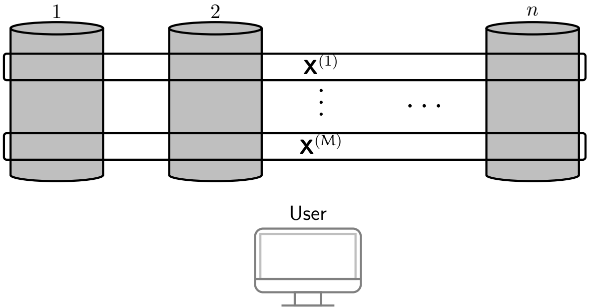
- linear-coded data:
  - using an arbitrary $[n, k]$ linear code to encode the data
  - LRCs are (in general) not MDS codes

Private Information Retrieval (PIR) for Distributed Storage

- n non-colluding servers: $l \in [1:n]$
- M files: $\mathbf{X}^{(m)}$, $m \in [\mathrm{M}]$
- file size: β symbols



PIR: Upload

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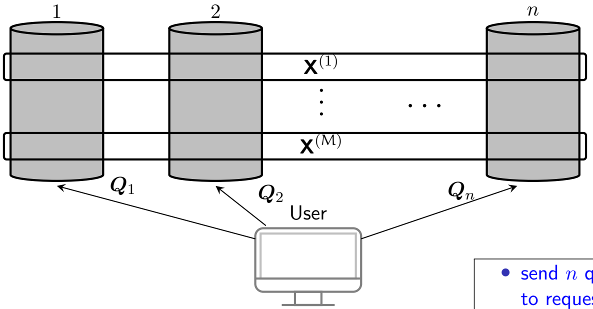

- send n queries $\boldsymbol{Q}_l$ to request $M \in [\mathrm{M}]$
- M : uniformly distributed

PIR: Download

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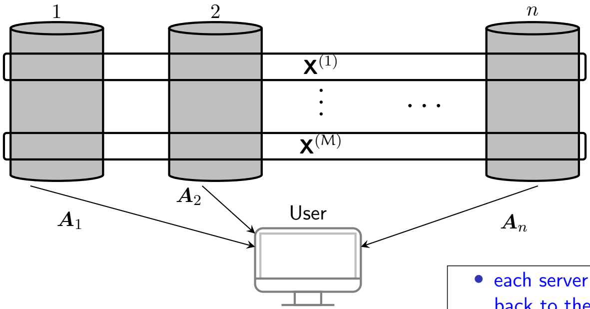

- each server sends $\boldsymbol{A}_l$ back to the user

Requirements of PIR

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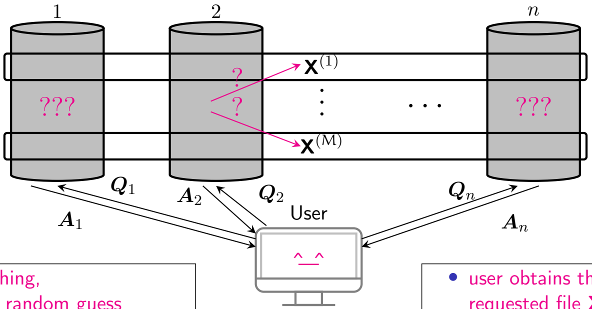

- servers learn nothing, only can make a random guess

- user obtains the complete requested file $\mathbf{X}^{(M)}$

Information-Theoretic PIR

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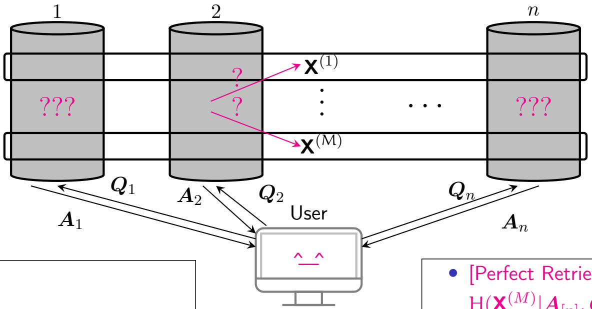

- [Strong Privacy]
  $M \perp\!\!\!\perp (\boldsymbol{Q}_l, \boldsymbol{A}_l)$

- [Perfect Retrievability]
  $\mathrm{H}(\mathbf{X}^{(M)} | \boldsymbol{A}_{[n]}, \boldsymbol{Q}_{[n]}, M) = 0$

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- PIR schemes were firstly studied in the computer science community
- PIR for replicated data (2 servers) was proposed
  - in the case of a single server, the solution is to download the entire data
- The efficiency of a classical PIR scheme is measured by the total amount of communication, i.e., the sum of the upload and download cost

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only the download cost is considered!!!

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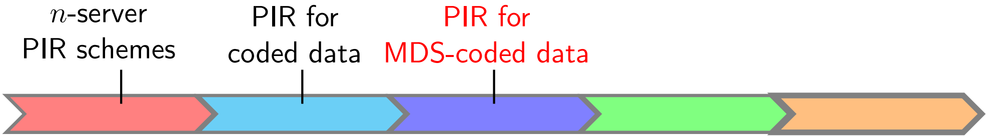

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- A practical PIR scheme for DSSs was proposed, where the data is encoded by an $[n, k]$ maximum distance separable (MDS) code $\mathscr{C}_{\text{MDS}}$

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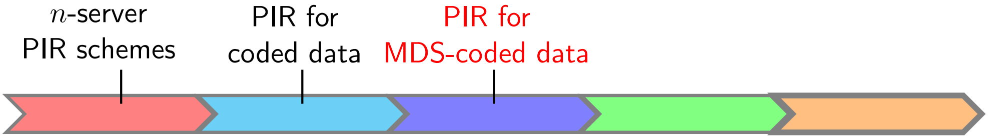

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- The PIR rate is independent of the number of files: a file-independent scheme

$$\mathrm{R}(\mathscr{C}_{\text{MDS}}) = 1 - \frac{k}{n}$$

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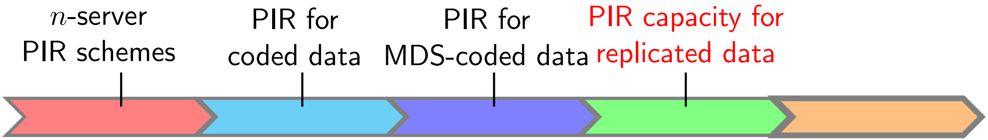

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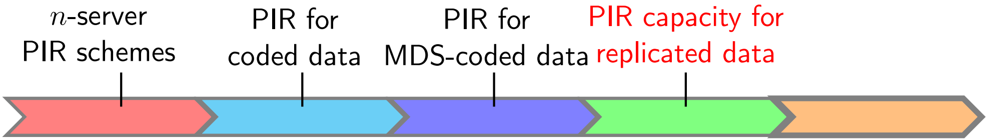

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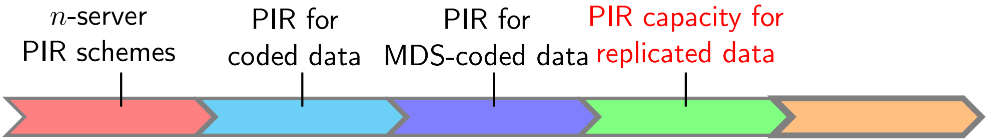

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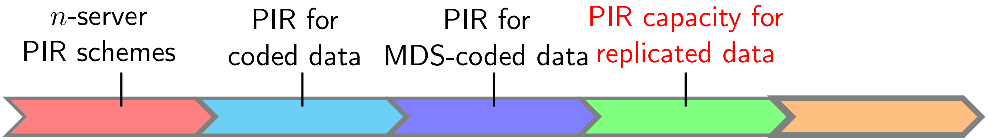

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$$\mathsf{C}_{\mathrm{M},\mathsf{PIR}} \triangleq \frac{1-\frac{1}{n}}{1-\left(\frac{1}{n}\right)^{\mathrm{M}}}$$

Channel Capacity vs. PIR Capacity

- **Channel capacity.** $P_{\mathsf{e}} \to 0$ as the code blocklength $n \to \infty$
  - Achievability: What is the transmission rate that a coding scheme can achieve (lower bound)?
  - Converse: What is the maximum possible transmission rate that a coding scheme can achieve (upper bound)?
- **PIR capacity.** Ensure privacy and correctness (zero error) as the file size $\beta \to \infty$
  - Achievability: What is the PIR rate that a PIR scheme can achieve (lower bound)?
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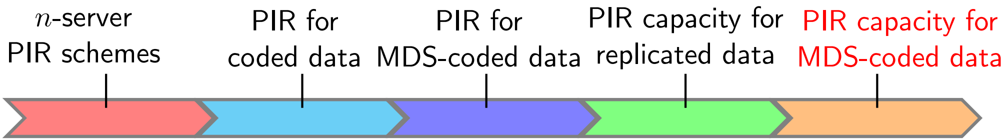

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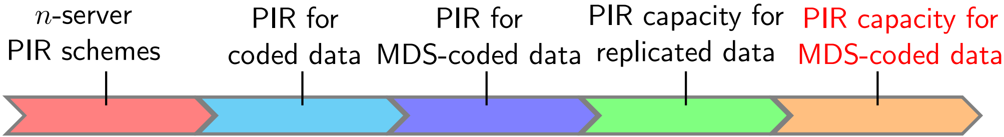

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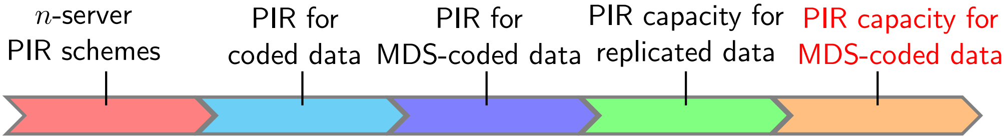

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PIR rate of the file-independent scheme proposed by Tajeddine and El Rouayheb

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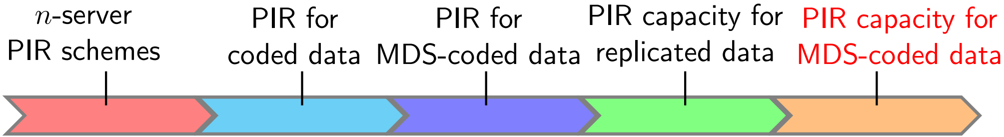

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asymptotic MDS-PIR capacity

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A PIR Scheme Using Subpacketization [Shah et al. 2014]: n Servers, $\beta = n-1$

Data is encoded by an $[n,1]$ repetition code with $\beta = n-1$

Server 1	Server 2	$\cdots$	Server n
$\mathbf{X}^{(1)}$	$\mathbf{X}^{(1)}$	$\cdots$	$\mathbf{X}^{(1)}$
$\mathbf{X}^{(2)}$	$\mathbf{X}^{(2)}$	$\cdots$	$\mathbf{X}^{(2)}$
$\vdots$	$\vdots$	$\cdots$	
$\mathbf{X}^{(m)}$	$\mathbf{X}^{(m)}$	$\cdots$	$\mathbf{X}^{(m)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{X}^{(M)}$	$\mathbf{X}^{(M)}$	$\cdots$	$\mathbf{X}^{(M)}$

$\mathbf{X}^{(m)}$: $\beta = (n-1)\cdot 1$

$X^{(m)}_{1,1}$
$X^{(m)}_{2,1}$
$\vdots$
$X^{(m)}_{n-1,1}$

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- To retrieve $\mathbf{X}^{(m)} = \left[X_{1,1}^{(m)}, \ldots, X_{n-1,1}^{(m)}\right]^{\mathsf{T}}$ of size $n - 1$

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- Pick a random vector $\boldsymbol{U} = \bigl(U_{1,1}, \ldots, U_{1,n-1}, U_{2,1}, \ldots, U_{\mathsf{M},(n-1)}\bigr)$ with i.i.d. entries $\sim \mathsf{Uniform}(\mathsf{GF}(2))$

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- Queries: Send $\boldsymbol{U}$ to Server 1, send $\boldsymbol{U} + \boldsymbol{e}_{(m-1)\cdot(n-1)+1}$ to Server 2,..., and send $\boldsymbol{U} + \boldsymbol{e}_{(m-1)\cdot(n-1)+(n-1)}$ to Server n

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The PIR Capacity-Achieving Scheme for Replicated Data

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  - a Shannon-theoretic re-formulation of converse bound is proposed

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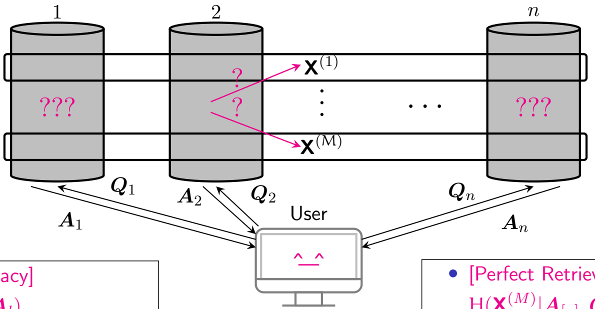
  - those codes are referred to as **MDS-PIR capacity-achieving codes**
  - a generalization of the Sun-Jafar scheme and Banawan-Ulukus scheme

Review: Information-Theoretic PIR



- [Strong Privacy]
  $M \perp\!\!\!\perp (\boldsymbol{Q}_l, \boldsymbol{A}_l)$

- [Perfect Retrievability]
  $\mathrm{H}(\mathbf{X}^{(M)} | \boldsymbol{A}_{[n]}, \boldsymbol{Q}_{[n]}, M) = 0$

Retrievability: Use Information Sets

$X^{(m)}: \ \beta = \beta' \times k$

$$\beta' \left\{ \underbrace{\begin{array}{|c|c|c|c|} \hline X^{(m)}_{1,1} & X^{(m)}_{1,2} & \cdots & X^{(m)}_{1,k} \\ \hline X^{(m)}_{2,1} & X^{(m)}_{2,2} & \cdots & X^{(m)}_{2,k} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline X^{(m)}_{\beta',1} & X^{(m)}_{\beta',2} & \cdots & X^{(m)}_{\beta',k} \\ \hline \end{array}}_{k} \right.$$

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- **Information set:** a set $\mathcal{I} = \{j_1, \ldots, j_k\} \subseteq [1 : n]$ such that the code symbols $(C_{j_1}, C_{j_2}, \ldots, C_{j_k})$ determine the k information symbols

MDS-PIR Capacity-Achieving Codes

- For any $[n, k]$ code $\mathscr{C}$, one can always find two parameters ν and κ such that
  - **each coordinate $j \in [1:n]$ appears exactly κ times in ν sets $\{\mathcal{S}_i\}_{i=1}^{\nu}$**
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- $\mathscr{C}_{\mathsf{MDS}} \subset \mathscr{C}^* : (\nu, \kappa) = (n, k)$

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- **Theorem. The PIR capacity for MDS-PIR capacity-achieving codes is equal to the MDS-PIR capacity $\mathsf{C}_{\mathsf{M}}^{[n,k]}$**

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- Only the PIR capacity of non-MDS-PIR capacity-achieving codes is not yet determined!

Example: An MDS-PIR Capacity-Achieving $[5, 3, 2]$ Code

- Consider a $[5, 3, 2]$ binary non-MDS code $\mathscr{C}$ with generator matrix

$$\mathsf{G} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

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- There exist 5 information sets of $\mathscr{C}$ such that $((\nu, \kappa) = (5, 3))$

information sets	code coordinates				
$\mathcal{S}_1 = \mathcal{I}_1$	1	2	3		
$\mathcal{S}_2 = \mathcal{I}_2$	1			4	5
$\mathcal{S}_3 = \mathcal{I}_3$		2		4	5
$\mathcal{S}_4 = \mathcal{I}_4$		2	3	4	
$\mathcal{S}_5 = \mathcal{I}_5$	1		3		5

Stack

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$$\frac{\kappa}{\nu} = \frac{k}{n} = \frac{3}{5}$$

Example: A Non-MDS-PIR Capacity-Achieving $[5, 3, 2]$ Code

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information sets	sets	code coordinates				
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Stack

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Stack $\longrightarrow$

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$(\nu, \kappa) = (3, 2)$

Symmetric vs. Asymmetric PIR Protocols

- Consider the binary $[5, 3, 2]$ non-MDS-PIR capacity-achieving code $\mathscr{C}'$
- Define the interference symbols: $I_{h,l} \triangleq \boldsymbol{u}_h^{\mathsf{T}} \boldsymbol{c}_l$, $h \in \mathbb{N}$, $l \in [1 : n]$
- $\boldsymbol{u}$ is a vector of length βM with i.i.d. uniformly distributed components

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- **Symmetric Protocol:** $\mathsf{H}(A_1) = \cdots = \mathsf{H}(A_5) = 2$ $\qquad$ $\mathsf{G}' = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$

Responses	Server 1	Server 2	Server 3	Server 4	Server 5
$\mathsf{H}(A_l)$	$I_{1,1} + x_{1,1}^{(m)}$	$I_{1,2}$	$I_{1,3}$	$I_{1,4}$	$I_{1,5}$
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- $\mathcal{S}_3' = \{1,2,3\} \Longrightarrow$ recover the m-th file: $\left[x_{1,1}^{(m)}, x_{1,2}^{(m)}, x_{1,3}^{(m)}\right]$ $(\beta = 1)$

Symmetric vs. Asymmetric PIR Protocols

- Theorem (Symmetric Protocol). The PIR rate

$$\mathrm{R}_{\mathrm{M},\,\mathrm{S}}(\mathscr{C}) = \frac{(\nu-\kappa)k}{\kappa n}\left[1-\left(\frac{\kappa}{\nu}\right)^{\mathrm{M}}\right]^{-1} \quad \text{is achievable}$$

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Asymmetry Helps for Non-MDS-PIR Capacity-Achieving Codes

- **Theorem (Asymmetric Protocol A). The PIR rate**

$$\mathrm{R}_{\mathrm{M},\,\mathrm{A}}(\mathscr{C}) \triangleq \left(1-\frac{\kappa}{\nu}\right)\left[1-\left(\frac{\kappa}{\nu}\right)^{\mathrm{M}}\right]^{-1} \quad \text{is achievable}$$

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- It can be shown that $\mathrm{R}_{\mathrm{M,\,S}}(\mathscr{C}) < \mathrm{R}_{\mathrm{M,\,A}}(\mathscr{C})$ for any given ν and κ

Asymmetric Protocol for a Special Class of Codes

- Consider the $[5, 3, 2]$ binary non-MDS code $\mathscr{C}'$ with generator matrix

$$\mathsf{G}' = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

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- Decompose G' into $\mathsf{G}'_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathsf{G}'_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$:

Protocol A:

Responses	Server 1	Server 2	Server 3	Server 4	Server 5
$\mathsf{H}(A_l)$	$I_{1,1} + x_{1,1}^{(m)}$	$I_{1,2}$	$I_{1,3}$	$I_{1,4}$	$I_{1,5}$
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Protocol B:

Responses	Server 1	Server 2	Server 3	Server 4	Server 5
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Asymmetric Protocol for a Special Class of Codes

- If the generator matrix G of a non-MDS-PIR capacity-achieving code $\mathscr{C}_{\mathsf{DS}}$ has the structure $(n = \sum_{p=1}^{\mathsf{P}} n_p, k = \sum_{p=1}^{\mathsf{P}} k_p$, G_p: size $k_p \times n_p)$

$$\mathsf{G} = \begin{pmatrix} \mathsf{G}_1 & & & \\ & \mathsf{G}_2 & & \\ & & \ddots & \\ & & & \mathsf{G}_{\mathsf{P}} \end{pmatrix}, \quad \mathscr{C}^{\mathsf{G}_p} : [n_p, k_p] \text{ MDS-PIR capacity-achieving codes with } \mathsf{G}_p$$

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- **Theorem (Asymmetric Protocol B). The PIR rate**

$$\mathsf{R}_{\mathsf{M},\,\mathsf{B}}(\mathscr{C}_{\mathsf{DS}}) \triangleq \left(\sum_{p=1}^{\mathsf{P}} \frac{k_p}{k} \left(\mathsf{C}_{\mathsf{M}}^{[n_p,k_p]} \right)^{-1} \right)^{-1} \quad \text{is achievable}$$

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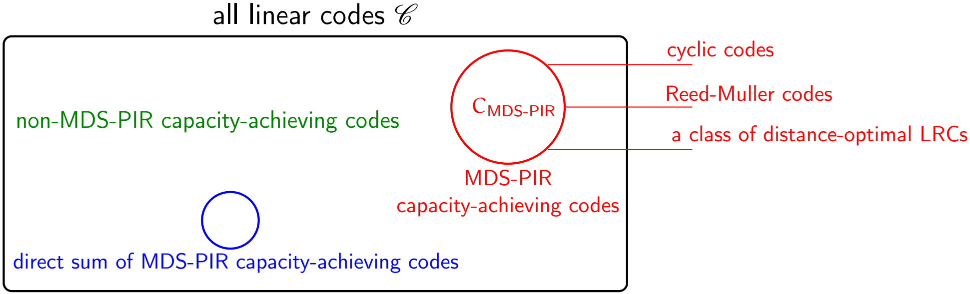
- If such code exists, then one can show that $\mathsf{R}_{\mathsf{M},\,\mathsf{A}}(\mathscr{C}_{\mathsf{DS}}) < \mathsf{R}_{\mathsf{M},\,\mathsf{B}}(\mathscr{C}_{\mathsf{DS}})$

Take Home Messages



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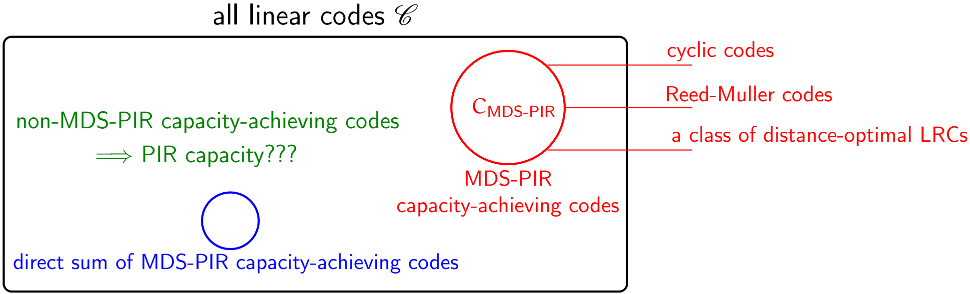
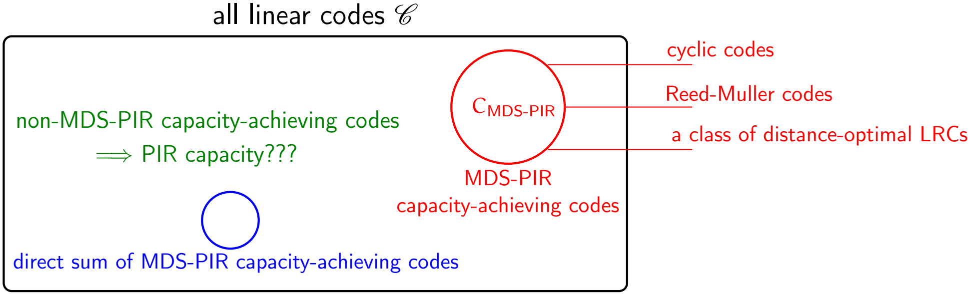


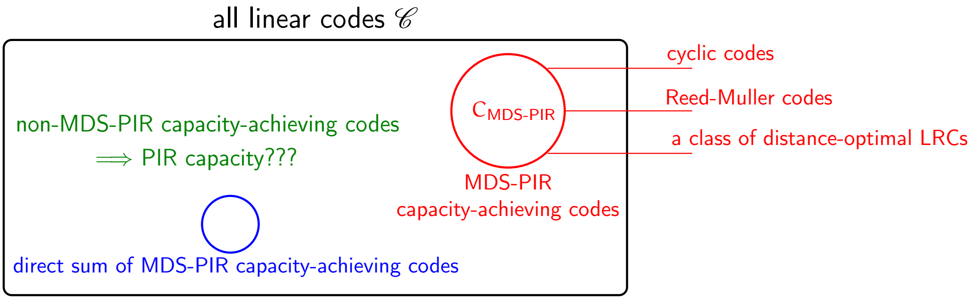
- For codes that cannot be decomposed into a direct sum of MDS-PIR capacity-achieving codes, the PIR capacity is still unknown...
  - a code-dependent, but file-independent asymmetric protocol is also proposed
  - Protocol A could still be improved

Take Home Messages



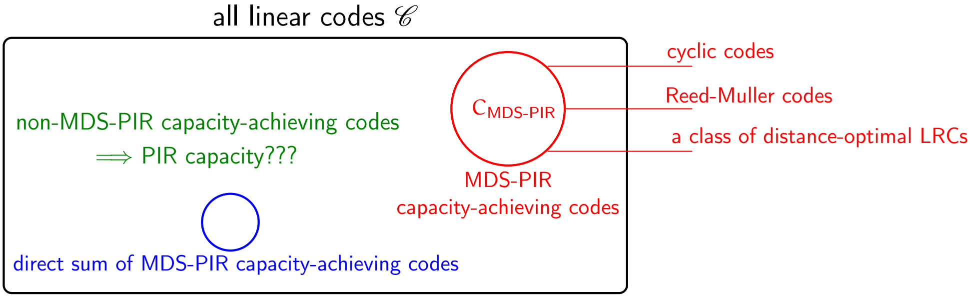
- **Open Problem 1: What is the PIR capacity for non-MDS encoded data?**

Take Home Messages



- **Open Problem 2: Is $\mathrm{C}_{\mathrm{M}}^{[n,k]}$ the limit for any $[n,k]$ linear-coded DSS?**
  - So far, no $[n,k]$ non-MDS-PIR capacity-achieving code gives a strictly larger PIR rate than $\mathrm{C}_{\mathrm{M}}^{[n,k]}$

Take Home Messages



- Asymmetric PIR protocols could be needed for a coded DSS using non-MDS-PIR capacity-achieving codes