Proof theory of skew non-commutative MILL

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Background

- Canonical model of non-commutative multiplicative intuitionistic linear logic (NMILL) is monoidal closed category.
- Skew monoidal closed category is a weaker version of monoidal closed category.
- In previous studies (Uustalu et al.' 18, 20), proof theoretical characterizations of skew monoidal categories and skew closed categories are developed. These systems are variants of NMILL.
- Question: is it possible to find a deductive system naturally modelled by skew monoidal closed categories?
- Results:
  - Construct a sequent calculus system $\mathtt{NMILL}^s$ which characterizes skew monoidal closed categories.
  - Solve the coherence problem of skew monoidal closed categories.

Skew monoidal closed category

- A *(left) skew monoidal closed category* $\mathbb{C}$ is a category with a unit object I and two functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ forming an adjunction $- \otimes B \dashv B \multimap -$ for all B, and three natural transformations λ, ρ, α typed

$$\lambda_A : \mathsf{I} \otimes A \to A$$
$$\rho_A : A \to A \otimes \mathsf{I}$$
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

  satisfying 5 Mac Lane axioms (Mac Lane'63)
- Coherence problem is about existence and uniqueness of morphisms in a category. Thanks to Curry-Howard-Lambek correspondence, we can use proof theoretical method to analyze categories.

Sequent calculus of skew monoidal closed categories: $\texttt{NMILL}^s$

- Formulae over a set At of atoms: $A, B ::= X \mid \mathsf{I} \mid A \otimes B \mid A \multimap B$.
- Sequents are triples $S \mid \Gamma \vdash C$ where
  - S is an optional formula,
  - Γ is an ordered list of formulae,
  - C is a single formula.
- Derivations are constructed with following inference rules:

$$\frac{}{A \mid \ \vdash A}\,\mathsf{ax} \qquad \frac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C}\,\mathsf{pass}$$

$$\frac{- \mid \Gamma \vdash C}{\mathsf{I} \mid \Gamma \vdash C}\,\mathsf{IL} \qquad \frac{}{- \mid \ \vdash \mathsf{I}}\,\mathsf{IR}$$

$$\frac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C}\,{\multimap}\mathsf{L} \qquad \frac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B}\,{\multimap}\mathsf{R}$$

$$\frac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C}\,{\otimes}\mathsf{L} \qquad \frac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B}\,{\otimes}\mathsf{R}$$

Why NMILL^s works: unitors

- A derivation corresponding to the unitor λ:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \mid\ \vdash A}\ \mathsf{ax}}{- \mid A \vdash A}\ \mathsf{pass}}{\mathsf{I} \mid A \vdash A}\ \mathsf{IL}}{\mathsf{I} \otimes A \mid\ \vdash A}\ \otimes\mathsf{L}$$

- No derivation corresponding to λ^{-1}:

$$\dfrac{\overset{??}{X \mid\ \vdash \mathsf{I}} \quad \overset{??}{- \mid\ \vdash X}}{X \mid\ \vdash \mathsf{I} \otimes X}\ \otimes\mathsf{R}$$

- A derivation corresponding to the unitor ρ:

$$\dfrac{\dfrac{}{A \mid\ \vdash A}\ \mathsf{ax} \quad \dfrac{}{- \mid\ \vdash \mathsf{I}}\ \mathsf{IR}}{A \mid\ \vdash A \otimes \mathsf{I}}\ \otimes\mathsf{R}$$

- No derivation corresponding to ρ^{-1}:

$$\dfrac{\overset{??}{X \mid \mathsf{I} \vdash X}}{X \otimes \mathsf{I} \mid\ \vdash X}\ \otimes\mathsf{L}$$

Why $\mathtt{NMILL}^s$ works: associator

- A derivation corresponding to the associator α:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\,\mathsf{ax} \qquad \dfrac{\dfrac{\dfrac{}{B \mid \ \vdash B}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{C \mid \ \vdash C}\,\mathsf{ax}}{- \mid C \vdash C}\,\mathsf{pass}}{B \mid C \vdash B \otimes C}\,\otimes\mathsf{R}}{- \mid B, C \vdash B \otimes C}\,\mathsf{pass}}{A \mid B, C \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{R}}{A \otimes B \mid C \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{L}}{(A \otimes B) \otimes C \mid \ \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{L}$$

- No derivation corresponding to α^{-1}:

$$\dfrac{\dfrac{??}{X \mid Y \otimes Z \vdash (X \otimes Y) \otimes Z}}{X \otimes (Y \otimes Z) \mid \ \vdash (X \otimes Y) \otimes Z}\,\otimes\mathsf{L}$$

Results

- Two forms of cut rules are admissible:

$$\frac{S \mid \Gamma \vdash A \quad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C}\ \mathsf{scut} \qquad \frac{- \mid \Gamma \vdash A \quad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C}\ \mathsf{ccut}$$

  So $\mathtt{NMILL}^s$ is a good proof theory.

- We define an equivalence relation $\circeq$ on derivations in $\mathtt{NMILL}^s$. For example,

$$\frac{}{\mathsf{I} \mid\ \vdash \mathsf{I}}\ \mathsf{ax} \qquad \circeq \qquad \frac{\frac{}{- \mid\ \vdash \mathsf{I}}\ \mathsf{IR}}{\mathsf{I} \mid\ \vdash \mathsf{I}}\ \mathsf{IL}$$

  $\mathtt{NMILL}^s$ with derivations quotiented by the equivalence relation $\circeq$ forms the free skew monoidal closed category over set At.

Focused sequent calculus: first attempt

(right invertible)

$$\frac{S \mid \Gamma, A \vdash_{\mathsf{RI}} B}{S \mid \Gamma \vdash_{\mathsf{RI}} A \multimap B} \multimap\mathsf{R} \qquad \frac{S \mid \Gamma \vdash_{\mathsf{LI}} P}{S \mid \Gamma \vdash_{\mathsf{RI}} P} \mathsf{LI2RI}$$

(left invertible)

$$\frac{- \mid \Gamma \vdash_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{LI}} P} \mathsf{IL} \qquad \frac{A \mid B, \Gamma \vdash_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P} \otimes\mathsf{L} \qquad \frac{T \mid \Gamma \vdash_{\mathsf{P}} P}{T \mid \Gamma \vdash_{\mathsf{LI}} P} \mathsf{P2LI}$$

(passivation)

$$\frac{A \mid \Gamma \vdash_{\mathsf{LI}} P}{- \mid A, \Gamma \vdash_{\mathsf{P}} P} \mathsf{pass} \qquad \frac{T \mid \Gamma \vdash_{\mathsf{F}} P}{T \mid \Gamma \vdash_{\mathsf{P}} P} \mathsf{F2P}$$

(focusing)

$$\frac{}{X \mid \ \vdash_{\mathsf{F}} X} \mathsf{ax} \qquad \frac{}{- \mid \ \vdash_{\mathsf{F}} \mathsf{I}} \mathsf{IR}$$

$$\frac{T \mid \Gamma \vdash_{\mathsf{RI}} A \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}} A \otimes B} \otimes\mathsf{R} \qquad \frac{- \mid \Gamma \vdash_{\mathsf{RI}} A \quad B \mid \Delta \vdash_{\mathsf{LI}} P}{A \multimap B \mid \Gamma, \Delta \vdash_{\mathsf{F}} P} \multimap\mathsf{L}$$

- P denotes a positive formula, i.e. $P \neq A \multimap B$.
- T denotes a negative stoup (possibly empty), i.e. $T \neq \mathsf{I}$ and $T \neq A \otimes B$.

Too much non-determinism

- There are two types of non-determinism show that the first attempt is too permissive.
- Type 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{f}{X \mid \Gamma \vdash_{\mathsf{LI}} P}}{X \mid \Gamma \vdash_{\mathsf{RI}} P}\ \mathsf{sw} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} C}}{X \mid \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C}\ {\otimes}\mathsf{R}}{X \mid \Gamma, \Delta \vdash_{\mathsf{LI}} P \otimes C}\ \mathsf{sw}}{- \mid X, \Gamma, \Delta \vdash_{\mathsf{P}} P \otimes C}\ \mathsf{pass}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\overset{f}{X \mid \Gamma \vdash_{\mathsf{LI}} P}}{- \mid X, \Gamma \vdash_{\mathsf{P}} P}\ \mathsf{pass}}{- \mid X, \Gamma \vdash_{\mathsf{RI}} P}\ \mathsf{sw} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} C}}{- \mid X, \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C}\ {\otimes}\mathsf{R}}{- \mid X, \Gamma, \Delta \vdash_{\mathsf{P}} P \otimes C}\ \mathsf{sw}$$

- Type 2:

$$\dfrac{\overset{f}{- \mid \Gamma \vdash_{\mathsf{RI}} A} \qquad \dfrac{\dfrac{\dfrac{\overset{g}{X \mid \Delta \vdash_{\mathsf{LI}} P}}{X \mid \Delta \vdash_{\mathsf{RI}} P}\ \mathsf{sw} \qquad \overset{h}{- \mid \Lambda \vdash_{\mathsf{RI}} D}}{X \mid \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\ {\otimes}\mathsf{R}}{X \mid \Delta, \Lambda \vdash_{\mathsf{LI}} P \otimes D}\ \mathsf{sw}}{A \multimap X \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\ {\multimap}\mathsf{L}
\qquad
\dfrac{\dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash_{\mathsf{RI}} A} \quad \overset{g}{X \mid \Delta \vdash_{\mathsf{LI}} P}}{A \multimap X \mid \Gamma, \Delta \vdash_{\mathsf{F}} P}\ {\multimap}\mathsf{L}}{A \multimap X \mid \Gamma, \Delta \vdash_{\mathsf{RI}} P}\ \mathsf{sw} \qquad \overset{h}{- \mid \Lambda \vdash_{\mathsf{RI}} D}}{A \multimap X \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\ {\otimes}\mathsf{R}$$

Motivation of focused sequent calculus with tag annotations

- Regarding to type 1 non-determinism, we want to put restrictions on pass application after $\otimes$R application (from bottom-up perspective). One possible way to do that is to check whether the leftmost formula in the context is new, i.e. it does not exist without the previous $\otimes$R application.
- Similarly, if we want to apply $\multimap$L after $\otimes$R, we need to check whether context Γ has any new formula in sequent $A \multimap B \mid \Gamma, \Delta \vdash_{\mathsf{F}} P$
- We use $\bullet$ to denote a new formula and $S \mid \Gamma \vdash^{\bullet}_{ph} C$ $(ph = \{\mathsf{RI}, \mathsf{LI}, \mathsf{P}, \mathsf{F}\})$ to denote a tagged sequent.
- A sequent becomes tagged in a derivation if it appears at the first premise of a $\otimes$R application. A formula is new if it is moved from succedent to context via $\multimap$R in a tagged sequent.

Focused sequent calculus with tag annotations

(right invertible)

$$\frac{S \mid \Gamma, A^x \vdash^x_{\mathsf{RI}} B}{S \mid \Gamma \vdash^x_{\mathsf{RI}} A \multimap B} \multimap\mathsf{R} \qquad \frac{S \mid \Gamma \vdash^x_{\mathsf{LI}} P}{S \mid \Gamma \vdash^x_{\mathsf{RI}} P} \mathsf{LI2RI}$$

(left invertible)

$$\frac{- \mid \Gamma \vdash_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{LI}} P} \mathsf{IL} \qquad \frac{A \mid B, \Gamma \vdash_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P} \otimes\mathsf{L} \qquad \frac{T \mid \Gamma \vdash^x_{\mathsf{P}} P}{T \mid \Gamma \vdash^x_{\mathsf{LI}} P} \mathsf{P2LI}$$

(passivation)

$$\frac{A \mid \Gamma^\circ \vdash_{\mathsf{LI}} P}{- \mid A^x, \Gamma \vdash^x_{\mathsf{P}} P} \mathsf{pass} \qquad \frac{T \mid \Gamma \vdash^x_{\mathsf{F}} P}{T \mid \Gamma \vdash^x_{\mathsf{P}} P} \mathsf{F2P}$$

(focusing)

$$\frac{}{X \mid \ \vdash^x_{\mathsf{F}} X} \mathsf{ax} \qquad \frac{}{- \mid \ \vdash^x_{\mathsf{F}} \mathsf{I}} \mathsf{IR}$$

$$\frac{T \mid \Gamma^\circ \vdash^\bullet_{\mathsf{RI}} A \quad - \mid \Delta^\circ \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash^x_{\mathsf{F}} A \otimes B} \otimes\mathsf{R}$$

$$\frac{- \mid \Gamma^\circ \vdash_{\mathsf{RI}} A \quad B \mid \Delta^\circ \vdash_{\mathsf{LI}} P \quad x = \bullet \supset \bullet \in \Gamma}{A \multimap B \mid \Gamma, \Delta \vdash^x_{\mathsf{F}} P} \multimap\mathsf{L}$$

The side condition in rule $\multimap$L reads: if $x = \bullet$, there exists an occurrence of a tagged formula $D^\bullet$ in Γ.

Turn equivalence into equality

- We can define a bijection function $\mathsf{focus} : S \mid \Gamma \vdash C \longrightarrow S \mid \Gamma \vdash_{\mathsf{RI}} C$ which sends $\overset{\circ}{=}$-equivalent derivations to syntactically identical derivations.
- Formally speaking, given $f, g : S \mid \Gamma \vdash C$, $f \overset{\circ}{=} g$ iff $\mathsf{focus}\ (f) = \mathsf{focus}\ (g)$.

Conclusions

- NMILL^{s}characterizes skew monoidal closed structures.
- Existence part of coherence problem is solved by NMILL^s, and uniqueness is solved by tagged focused sequent calculus.
- Results in this paper are formalized in proof assistant Agda: *https://github.com/niccoloveltri/code-skewmonclosed*