Stateful relators for algebraic effects

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Introduction

We study functional languages with the features:

- Stateful effects.
- Nondeterminism.
- General Recursion.

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Tools: framework for easy of formalisation in Agda:

- Programs modelled by inductive structures, trees.
- Programs equated using relational reasoning, relators.

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Algebraic Operations

Signature $S = (O, \mathtt{ar}) : \Sigma_{O:\mathsf{Set}} O \to \mathsf{Set}$
Each *operation* $\mathtt{op} : O$ has an arity $\mathtt{ar}(\mathtt{op}) : \mathsf{Set}$.

Inductive set $T_S X$ of *S-trees* (terms) over a set X:

- Leaves $\langle - \rangle : X \to T_S X$.
- For each $\mathtt{op} : O$, $\mathtt{op}(-) : (\mathtt{ar}(\mathtt{op}) \to T_S X) \to T_S X$.

If $\mathtt{ar}\ \mathtt{op} = \{1, \ldots, n\}$, we may write $\mathtt{op}(x_1, \ldots, x_n)$ for $\mathtt{op}(\lambda i.\, x_i)$.

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Kleisli lifting as binding operation, (sequencing):
$\kappa_{X,Y} : T_S X \to (X \to T_S Y) \to T_S Y$.

- $\kappa_{X,Y}(\langle x \rangle)(f) = f(x)$.
- $\kappa_{X,Y}(\mathtt{op}(c))(f) = \mathtt{op}(\lambda i.\ \kappa_{X,Y}(c(i))(f))$

Relators

A *relator* Γ on T_S lifts a relation $\mathcal{R} \subseteq X \times Y$ (on values) to a relation $\Gamma(\mathcal{R}) \subseteq T_S X \times T_S Y$ (on computations) such that:

(1) $=_{T_S X} \subseteq \Gamma(=_X)$

(2) $\Gamma(\mathcal{R})\Gamma(\mathcal{U}) \subseteq \Gamma(\mathcal{R}\mathcal{U})$.

(3) $\mathcal{R} \subseteq \mathcal{U} \implies \Gamma(\mathcal{R}) \subseteq \Gamma(\mathcal{U})$.

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(3) $\mathcal{R} \subseteq \mathcal{U} \implies \Gamma(\mathcal{R}) \subseteq \Gamma(\mathcal{U})$. (4) *naturality*

We seek relators Γ on T_S which are *monadic*:

- (*Unital*) $x \,\mathcal{R}\, y \implies \langle x \rangle \,\Gamma(\mathcal{R})\, \langle y \rangle$.
- (*Sequential*) Given $a \,\Gamma(\mathcal{R})\, b$ and $(\forall xy.\, x \,\mathcal{R}\, y \Rightarrow f(x) \,\Gamma(\mathcal{U})\, g(y))$, then $\kappa(a)(f) \,\Gamma(\mathcal{U})\, \kappa(b)(g)$.

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Minimal monadic relator Δ^S on T_S, inductively:

- If $x \, \mathcal{R} \, y$ then $\langle x \rangle \, \Delta^S(\mathcal{R}) \, \langle y \rangle$.
- For $\mathtt{op} : O$, if $\forall i : \mathtt{ar}(\mathtt{op}). \, c(i) \, \Delta^S(\mathcal{R}) \, d(i)$, then $\mathtt{op}(c) \, \Delta^S(\mathcal{R}) \, \mathtt{op}(d)$,

Nondeterminism

We model finite powerset using S_{nd} with two operations:

- $\oplus$ with arity $\{0, 1\}$, for binary nondeterministic choice.
- ε with arity $\emptyset$, for nontermination or error.

Relator Γ_{nd} on $P = T_{S_{nd}}$ where $a \, \Gamma_{nd}(\mathcal{R}) \, b$ holds if for any leaf $\langle x \rangle$ of a, there is a leaf $\langle y \rangle$ of b such that $x \, \mathcal{R} \, y$.

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Inductively:

- If $x\,\mathcal{R}\,y$ then $\langle x\rangle\,\Gamma_{nd}(\mathcal{R})\,\langle y\rangle$.
- If $\exists i \in \{0,1\}.\,\langle x\rangle\,\Gamma_{nd}(\mathcal{R})\,d_i$, then $\langle x\rangle\,\Gamma_{nd}(\mathcal{R})\,\oplus(d_0, d_1)$.
- If $\forall i \in \{0,1\}.\,c_i\,\Gamma_{nd}(\mathcal{R})\,b$, then $\oplus(c_0, c_1)\,\Gamma_{nd}(\mathcal{R})\,b$.
- $\varepsilon\,\Gamma_{nd}(\mathcal{R})\,b$.

$\Gamma_{nd}(=_X)$ simulates the subset relation on finite subsets of X.

Stateful effects

Take some signature S.

$\mathtt{op}(c)$ requests data of type $\mathtt{ar}(\mathtt{op})$ from the environment, which is given as an argument to the continuation c.

Consider an environment with internal state set K.

A (*nondeterministic stateful*) S-*runner* θ on K is given by:

- For each $\mathtt{op} : O$ a function $\theta_{\mathtt{op}} : K \to P(\mathtt{ar}(\mathtt{op}) \times K)$.

The leaves of $\theta_{\mathtt{op}}(m)$ give possible reactions (i, n):

- $i : \mathtt{ar}(\mathtt{op})$ is the piece of data.
- $n : K$ is the new internal state.

Examples

Input:

- One operation `input` of arity D.
- State set $D^{\mathbb{N}}$, infinite stream of data points.
- $\theta_{\texttt{input}}(f) = \langle (f(0), \lambda n.\, f(n+1)) \rangle$.

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Output:

- For each $d \in D$ an operation $\texttt{output}(d)$ of arity $\{*\}$.
- State set D^*, finite lists of printed data points.
- $\theta_{\texttt{output}(d)}(l) = \langle (*, d :: l) \rangle$

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Costly nondeterminism:

- An operation $\oslash$ of arity $\{0, 1\}$.
- State set $\mathbb{N}$, how much credit we have.
- $\theta_{\oslash}(0) = \{(0, 0)\}, \quad \theta_{\oslash}(n+1) = \oplus(\langle (0, n+1) \rangle, \langle (1, n) \rangle)$.

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Sequentiality

An S-runner θ on K gives a natural transformation:

$$\theta_X : T_S X \times K \to P(X \times K)$$

- $\theta_X(\langle x \rangle, m) = \langle (x, m) \rangle$.
- $\theta_X(\mathtt{op}(c), m) = \kappa_X^{nd}(\theta_{\mathtt{op}}(m))(\lambda(i, n).\, \theta_X(c(i), n))$.

Given a program $t \in T_S X$ and an initial state $m \in K$, then $\theta_X(t, m)$ represents a finite set of results of running t with state m, each result containing a value $x \in X$ produced by t and a final state $n \in K$.

This is *Sequential*:
$\theta_Y(\kappa_{X,Y}(t)(f), m) = \kappa_X^{nd}(\theta_X(t, m))(\lambda(x, n).\, \theta_Y(f(x), n))$

Global Relator

For $A \in PX$ and $x \in X$, we write $x \in X$ for: A has a leaf $\langle x \rangle$.

Suppose given an S-runner θ on K,
the global relator Γ^{θ} on T_S given by θ is:

$$a\,\Gamma^{\theta}(\mathcal{R})\,b \quad \iff \quad \forall m \in M.\, \theta_X(a, m)\,\Gamma_{nd}(\mathcal{R} \times =_K)\,\theta_X(a, m)$$

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In other words: For any initial state $m \in K$, for any $(x, n) \in \theta_X(a, m)$, there is a $y \in Y$ such that $(y, n) \in \theta_Y(b, m)$ and $x \, \mathcal{R} \, y$.

Proposition: This relator is monadic.

Example: For the input example, Γ^θ is the minimal monadic relator Δ^S.

Fine tuning I: State Observation

Suppose θ is an S-runner on K.

Which states are *separable*? What can be observed?

A preorder $\mathcal{R} \subseteq K \times K$ is θ-*closed* if:
$\forall \mathtt{op} : O,\, u \,\mathcal{R}\, v,\, (i, u') \in \theta_{\mathtt{op}}(u),\, \exists v' \in K$ s.t. $(i, v') \in \theta_{\mathtt{op}}(v)$.

By extension, $\forall t \in T_S X,\, u \,\mathcal{R}\, v \Rightarrow \theta_X(t, u)\, \Gamma_{nd}(=_X \times \mathcal{R})\, \theta_Y(t, v)$.

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By extension, $\forall t \in T_S X, u \,\mathcal{R}\, v \Rightarrow \theta_X(t, u) \,\Gamma_{nd}(=_X \times \mathcal{R})\, \theta_Y(t, v)$.

For any preorder $\mathcal{R} \subseteq K \times K$, we can take the largest θ-closed preorder contained in R, denoted $\mathcal{R}^\theta$.

We fix a θ-closed preorder. Canonical choices:

- Largest is $(K \times K)^\theta$.
- Smallest is $=_K$.

Fine tuning II: State Worlds

Which states can occur?

Consider the future relation $\rightsquigarrow \subseteq K \times K$ where
$u \rightsquigarrow v$ if $\exists t \in T_S X.\, x \in X.\, (x, v) \in \theta_X(t, u)$.

Lemma: This is the smallest preorder $\mathcal{R} \subseteq K \times K$ closed under: If $\exists \mathtt{op} : O.\, \exists i : \mathtt{ar}(\mathtt{op}).\, (i, v) \in \theta_{\mathtt{op}}(u)$, then $u \,\mathcal{R}\, v$.

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A *state world* is a set $W \subseteq K$ such that:
$u \in W \wedge u \rightsquigarrow v \;\Rightarrow\; v \in W$.

For $s \in K$, let $[s] \subseteq K$ be the smallest state world with s.

Stateful Relators

Let $\prec$ be a θ-closed preorder on K.

- For each $s \in K$, a relator Γ_s on T_S such that:
  $a \, \Gamma_s(\mathcal{R}) \, b \quad \text{if} \quad \theta_X(a, s) \, \Gamma_{nd}(\mathcal{R} \times \prec) \, \theta_Y(b, s)$.
- For each $W \subseteq K$, a relator Γ_W on T_S such that:
  $a \, \Gamma_W(\mathcal{R}) \, b \quad \text{if} \quad \forall s \in W. \, a \, \Gamma_s(\mathcal{R}) \, b$.

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- For each $W \subseteq K$, a relator Γ_W on T_S such that:
  $a\,\Gamma_W(\mathcal{R})\,b$ if $\forall s \in W.\, a\,\Gamma_s(\mathcal{R})\,b$.

If W is a state world containing s, then:

$$\frac{a\,\Gamma_s(\mathcal{R})\,b \qquad (\forall xy.\, x\,\mathcal{R}\,y \Rightarrow f(x)\,\Gamma_W(\mathcal{U})\,g(y))}{\kappa(a)(f)\,\Gamma_s(\mathcal{U})\,\kappa(b)(g)}$$

Theorem: For any state world W, Γ_W is monadic.

Examples

Take $(- \sqsubseteq_W -) \subseteq T_S X \times T_S X$ as $\Gamma_W(=_X)$.

Input:

- Take $=_{D^{\mathbb{N}}}$, which is the largest θ-closed preorder.
- A set $W \subseteq D^{\mathbb{N}}$ is a state world if $f \in W \Rightarrow \lambda n.f(n+1) \in W$.
- Let $D = \{0,1\}$, $\overline{01} = \lambda n.(n \mod 2) \in \{0,1\}^{\mathbb{N}}$, then $\mathtt{input}(\mathtt{input}(x,y),\mathtt{input}(y,z)) \sqsubseteq_{[\overline{01}]} y$.

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Nondeterminism + Cost:

- Largest θ-closed relation is $\leq$ on $\mathbb{N}$.
- A set $W \subseteq \mathbb{N}$ is a state world if $n+1 \in W \Rightarrow n \in W$.
- $\oslash(x,y) \sqsubseteq_{[0]} x, \quad x \sqsubseteq_{[n]} \oslash(x,y)$, $\oslash(x, \oslash(y,z)) \sqsubseteq_{[n]} \oslash(\oslash(x,y),z)$.

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Coinductive Trees

How to deal with recursion? Solution I: Coinduction.

Given a signature S, the set of coinductive trees $T'_S X$ over X is coinductive defined as: For any $t \in CT_S X$, either:

- $t = \langle x \rangle$ for some $x \in X$.
- or $t = \mathtt{op}(c)$ for some $\mathtt{op} : O$ and $c : \mathtt{ar}(\mathtt{op}) \to T'_S X$.

Let $P' = T'_{S_{nd}}$.

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Let $P' = T'_{S_{nd}}$.

Given a predicate $Q : X \to \mathsf{Set}$, we define two predicates $\Diamond Q, \Box Q : P'X \to \mathsf{Set}$ where:

- $\Diamond Q$ is inductively defined such that $(\Diamond Q)(t)$ holds if there is a leaf $\langle x \rangle$ such that x satisfies Q.
- $\Box Q$ is coinductively defined such that $(\Box Q)(t)$ holds if, any leaf $\langle x \rangle$ we can extract from t satisfies Q.

Relators for coinductive trees

For a relation $\mathcal{R} : X \to Y \to \mathsf{Set}$, $\mathcal{R}(x)$ is a predicate on Y.

Relator on P':
$a\, P'(\mathcal{R})\, b$ holds if $\Box(\lambda x.\Diamond(\mathcal{R}(x))(b))(a)$.

In other words: For any extractable leaf x from a, there is a leaf y from b such that $x\, \mathcal{R}\, y$.

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In other words: For any extractable leaf x from a, there is a leaf y from b such that $x\, \mathcal{R}\, y$.

Any S-runner θ on state set K also specifies a natural transformation coinductively:

$$\theta'_X : T'_S X \times K \to P'(X \times K)$$

Stateful relators could be extended?

Streams

How to deal with recursion? Solution II: Streams.

We add a bottom operation $\bot$ of arity $\emptyset$ to S, creating a new signature S'. Relation $\leq_X$ on $T_{S'}X$ inductively:

- $\langle x \rangle \ \leq_X \ \langle x \rangle$.
- For $\mathtt{op}$ from S, if $\forall i \in \mathtt{ar}(\mathtt{op}).\, c(i) \ \leq_X \ d(i)$, then $\mathtt{op}(c) \ \leq_X \ \mathtt{op}(d)$.
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- $\bot \ \leq_X \ b$.

We approximate an infinite tree using a stream of inductive trees $f : \mathbb{N} \to T_S X$, such that:

$$\forall n \in \mathbb{N}.\, f(n) \leq f(n+1)$$

Relators on Streams

An S-runner θ with state set K, can be extended uniquely to an S'-runner θ' on K, sending $\perp$ to ε.

For a stream $f : \mathbb{N} \to T_{S'}X$, $\theta'(f(n), m)\, \Gamma_{nd}(=)\, \theta'(f(n+1), m)$.

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Given $\mathcal{R} \subseteq X \times Y$, we define the relation $\Gamma'_W(\mathcal{R}) \subseteq (T_S X)^{\mathbb{N}} \times (T_S Y)^{\mathbb{N}}$ as:

$$f\,\Gamma'_W(\mathcal{R})\,g \iff \forall n \in \mathbb{N}.\,\exists m \in \mathbb{N}.\,f(n)\,\Gamma_W(\mathcal{R})\,g(m)$$

Conclusion

A formalism for modelling a variety of effects:

- Input/Output
- Nondeterminism
- Global store
- Cost
- Interleaving concurrency

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Tools for more fine-tuned stateful program equivalence:

- Notion of state relation for hiding unobservable aspects of the environment state.
- Notion of state world for reducing the possible initial environment states we need to check.

Bibliography

Based on work from:

- FSCD 2022, N.V.: Runners for concurrent effectful programs.
- MFPS 2021, Niccolò Veltri & N.V.: Inductive and Coinductive Predicate Liftings for Effectful Programs.
- Submitted to MPC 2022, Niccolò Veltri & N.V.: Streams of Approximations, Equivalence of Recursive effectful Programs.

Bonus example: Interleaving Concurrency

Suppose given an S-runner θ on K.

We can model running a program concurrently with some background process.
We take a new state set $T_S\{*\} \times K$, where $T_S\{*\}$ is some background process.

New S-runner:

$$\theta'_X : T_S X \times T_S\{*\} \times K \to \mathcal{P}(X \times T_S\{*\} \times K)$$

After handling the operation from the main process, the runner will handle any number of operations from the background process and keep the remainder.

Extra: Program Equivalence

Monadic relators give rise to *congruent* program equivalences for functional languages with general recursion and algebraic effects.

Idea: Computations are modelled by trees over values.
$|-| : C \to T_S V$

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- Approach I: Applicative similarity.
  Largest relation $\mathcal{R}$ on program terms, closed under application relator on computations.
- Approach II: Logical equivalence.
  Specify set of predicates for each type, and relate two programs if they satisfy the same predicates.
- Approach III: Denotations by Streams.
  Approximate programs with streams and define equivalence with relator.